DUALIZING INVOLUTIONS ON THE METAPLECTIC GL(2) *à la* TUPAN

KUMAR BALASUBRAMANIAN*, EKTA TIWARI

Abstract. Let F be a non-Archimedean local field of characteristic zero. Let $G = \mathrm{GL}(2, F)$ and $\widetilde{G} = \widetilde{\mathrm{GL}}(2, F)$ be the metaplectic group. Let τ be the standard involution on G. A well known theorem of Gelfand and Kazhdan says that the standard involution takes any irreducible admissible representation of G to its contragredient. In such a case, we say that τ is a dualizing involution. In this paper, we make some modifications and adapt a topological argument of Tupan to the metaplectic group $\widetilde{G}$ and give an elementary proof that any lift of the standard involution to $\widetilde{G}$ is also a dualizing involution.

1. Introduction

Let F be a non-Archimedean local field of characteristic 0 and $G = \mathrm{GL}(n, F)$. For $g \in G$, we let $g^\top$ denote the transpose of the matrix g, and w_0 to be the matrix with anti-diagonal entries equal to one. Let $\tau : G \to G$ be the map $\tau(g) = w_0 g^\top w_0$. It is easy to see that τ is an anti-automorphism of G such that $\tau^2 = 1$. We call τ the *standard involution* on G. Let (π, V) be an irreducible smooth complex representation of G. We write $(\pi^\vee, V^\vee)$ for the smooth dual or the contragredient of (π, V). For β an anti-automorphism of G such that $\beta^2 = 1$, we let π^β to be the twisted representation defined by

$$\pi^\beta(g) = \pi(\beta(g^{-1})).$$

The following theorem is an old result of Gelfand and Kazhdan.

Theorem 1.1 (Gelfand-Kazhdan). *Let τ be the standard involution on G. Then*

$$\pi^\tau \simeq \pi^\vee.$$

We refer the reader to Theorem 2 in [3] for a proof of the above result.

If β is any anti-automorphism of G such that $\beta^2 = 1$, and satisfies $\pi^\beta \simeq \pi^\vee$, then we call β a *dualizing involution*. The above result implies that the standard involution τ on G is a dualizing involution.

Let $\widetilde{G}$ be the metaplectic cover of G (see Chapter 0 in [5] for the general definition). It is well known that the standard involution τ on G has at least one lift to the metaplectic group (see Proposition 3.1 in [6]). A natural and interesting question that one can ask is whether the lifts of the standard involution are themselves dualizing involutions.

In an earlier work [1], we showed that this is true in the case when $\widetilde{G} = \widetilde{\mathrm{GL}}(2, F)$. The main idea in this work was to establish a crucial property of the lifts σ_α (see Theorem 5.6 in [1]) of τ to $\widetilde{G}$ and use the non-trivial fact that the character of the

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* indicates corresponding author.

representation is constant on (regular semisimple) conjugacy classes to establish the equality of the relevant distribution characters.

In [10], Tupan gives a very simple proof of the Gelfand-Kazhdan theorem for $G = \mathrm{GL}(2, F)$ by using an elementary topological argument (see Section 3, Lemmas 5, 6 and 7). In this paper, we make some slight modifications and adapt the topological argument of Tupan to the metaplectic group $\widetilde{G} = \widetilde{\mathrm{GL}}(2, F)$ and we use it to give a new and elementary proof of our earlier result.

We recall the statement of our main theorem below.

Theorem 1.2. *[Main Theorem] Let π be any irreducible admissible genuine representation of $\widetilde{G}$. For $\alpha \in F^\times$, let σ_α be the lift of τ to $\widetilde{G}$. Then*

$$\pi^{\sigma_\alpha} \simeq \pi^\vee.$$

The above result raises the question, if a similar statement can be established for the r-fold covering of $\mathrm{GL}(n)$ for $r, n \geq 2$. However, due to the complexity of the covering groups and the difficulty in explicit description of the lifts of the standard involution, we have so far not been able to prove the result in complete generality. Our hope is that the topological proof of Tupan can be adapted to resolve some of these questions. We plan to address them in the near future.

The paper is organized as follows. In Section 2, we recall a few preliminaries which we need. In Section 3, we explicitly describe the lifts of the standard involution and discuss an important property which we need to adapt Tupan's topological proof to the metaplectic setting. In Section 4, we prove the main result of this paper.

2. PRELIMINARIES

In this section, we set up the required preliminaries and recall a few results which we will need throughout this paper.

2.1. **Quadratic Hilbert Symbol and its properties.** Let F be a local field and $F^\times$ be the group of non-zero elements in F and let $\mu_2 = \{\pm 1\}$. The quadratic Hilbert symbol is a map

$$\langle\, ,\, \rangle : F^\times \times F^\times \to \mu_2$$

defined by

$$\langle a, b \rangle = \begin{cases} +1, & \text{if } z^2 - ax^2 - by^2 = 0 \text{ has a non-trivial solution in } F^3 \\ -1, & \text{otherwise} \end{cases}.$$

The following basic properties of the Hilbert symbol are well known. We record it in the proposition below.

Proposition 2.1. *The Hilbert symbol satisfies*

1) $\langle a, b \rangle = \langle b, a \rangle$ *and* $\langle a, c^2 \rangle = 1$.
2) $\langle a, -a \rangle = 1$ *and* $\langle a, 1 - a \rangle = 1$ *if* $a \neq 1$.
3) $\langle a, b \rangle = 1$ *implies* $\langle aa', b \rangle = \langle a', b \rangle$.
4) $\langle a, b \rangle = \langle a, -ab \rangle = \langle a, (1 - a)b \rangle$.
5) $\langle a, b \rangle = 1$ *for all* $a \in F^\times$, *then* $b \in (F^\times)^2$.

We refer the reader to Chapter 3, Section 1 in [9] for the details.

2.2. **Metaplectic Groups.** In this section, we define "the" metaplectic group $\widetilde{G}$ and recall a few basic facts.

Throughout, we write $\mathfrak{o}$ for the ring of integers in F, $\mathfrak{p}$ for the unique maximal ideal in $\mathfrak{o}$ and ϖ for the generator of $\mathfrak{p}$. We write k_F for the finite residue field and assume throughout that $char(k_F) \neq 2$. We write val for the valuation on F. The valuation is normalized such that $val(\varpi) = 1$.

The first explicit construction of a metaplectic cover of $\mathrm{GL}(2, F)$ was given by Kubota in [7] by concretely describing a 2-cocyle. For $g_1, g_2, m \in G$, the following simpler version of the Kubota cocycle $c : G \times G \to \mu_2$ defined as

$$c(g_1, g_2) = \left\langle \frac{X(g_1g_2)}{X(g_1)}, \frac{X(g_1g_2)}{X(g_2)\Delta(g_1)} \right\rangle,$$

where

$$X(m) = \begin{cases} m_{21} & \text{if } m_{21} \neq 0 \\ m_{22} & \text{otherwise} \end{cases}$$

was given by Kazhdan and Patterson in [6]. We take $\widetilde{G}$ to be the central extension of G by μ_2 determined by 2-cocycle c. Since G, μ_2 are locally compact groups, Mackey's theorem (see Theorem 2 in [8]) implies that $\widetilde{G}$ is a locally compact topological group and defines a topological central extension of G by μ_2. The group $\widetilde{G}$ constructed above is called"the" metaplectic group.

It can be shown that the topology on $\widetilde{G}$ has a neighborhood base at the identity consisting of compact open subgroups (see Lemma 3 in [4]). Before we give the construction of this basis, we recall a few preliminaries.

A map $\ell : G \to \widetilde{G}$ is called a *section* if $p \circ \ell = 1_G$ where $p : \widetilde{G} \to G$ is the natural projection map. Given a subgroup H of G, we say that $\widetilde{G}$ *splits* over H if there exists a homomorphism $h : H \to \widetilde{G}$ such that $p \circ h = 1_H$.

Let $\ell : G \to \widetilde{G}$ be the map $\ell(g) = (g, 1)$. Then ℓ is a section and is called the natural or preferred section. For $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$, let $\Delta(g) = \det(g)$ and define $s : G \to \mu_2$ as

$$s(g) = \begin{cases} \langle c, d\Delta(g) \rangle, & \text{if } cd \neq 0 \text{ and } val(c) \text{ is odd} \\ 1, & \text{otherwise.} \end{cases} \tag{2.1}$$

Let $K = \mathrm{GL}(2, \mathfrak{o})$ be the maximal compact subgroup in G, and for $\lambda \geq 1$, let $K_\lambda = 1 + \varpi^\lambda \, \mathrm{M}(n, \mathfrak{o})$. It is known that $\{K_\lambda\}_{\lambda \geq 1}$ is a neighborhood base at the identity element in G consisting of compact open subgroups. We can use this base to define a neighborhood base at the identity in $\widetilde{G}$. Define $\kappa : K \to \widetilde{G}$ as $\kappa(k) = (k, s(k))$. It can be shown that $\kappa : K \to \widetilde{G}$ is a homomorphism such that $p \circ \kappa = 1_K$, (i.e., $\widetilde{G}$ splits over K). Let $K^* = \kappa(K)$ and for $\lambda \geq 1$, $K^*_\lambda = K^* \cap p^{-1}(K_\lambda)$. It can be shown that $\{K^*_\lambda\}_{\lambda \geq 1}$ is a neighborhood base at the identity in $\widetilde{G}$.

2.3. **Distribution character of an admissible representation.** Let F be a non-Archimedean local field of characteristic 0 and $G = G(F)$ be a connected reductive algebraic group defined over F. We let (π, V) be an irreducible smooth complex representation of G. It can be shown that such representations are always admissible. For an admissible representation (π, V), we can define the notion of a distribution character. We recall it below for clarity.

Throughout we let $G = G(F)$ and (π, V) to be an irreducible smooth representation of G. We let $C_c^\infty(G)$ to be the space of all locally constant complex valued functions on G with compact support. For $f \in C_c^\infty(G)$, we let $\pi(f) : V \to V$ denote the linear operator given by

$$\pi(f)v = \int_G f(g)\pi(g)v dg, \quad v \in V,$$

where the integral is with respect to a Haar measure on G which we fix throughout. If (π, V) is an admissible representation, it can be shown that the trace of the operator $\pi(f)$ is finite for all $f \in C_c^\infty(G)$. The resulting linear functional

$$\Theta_\pi : C_c^\infty(G) \longrightarrow \mathbb{C}$$

given by

$$\Theta_\pi(f) = \mathrm{Tr}(\pi(f))$$

is called the *distribution character* of π. It determines the irreducible representation π up to equivalence, i.e., if $\Theta_{\pi_1}(f) = \Theta_{\pi_2}(f)$, $\forall f \in C_c^\infty(G)$, then $\pi_1 \simeq \pi_2$.

Let $\widetilde{G}$ be a locally compact topological central extension of G by μ_2, where μ_2, is the group of square roots of unity in F. Let $\xi : \mu_2 \to \mathbb{C}^\times$ be the non-trivial character of μ_2. Let (π, V) be an irreducible admissible representation of $\widetilde{G}$. π is called a *genuine* representation, if for $\epsilon \in \mu_2$, $g \in \widetilde{G}$, we have

$$\pi(\epsilon g) = \xi(\epsilon)\pi(g).$$

The above notion of the distribution character also makes sense when π is a genuine admissible representation of the covering group (see Section I.5 in [6]).

2.4. **Some known results about lifts of the standard involution.** We recall a few results from [4] which we need in proving our main result. Let $\widetilde{G}$ be a central extension of a group G by an abelian group A. Let $p : \widetilde{G} \to G$ be the projection map, $s : G \to \widetilde{G}$ be a section of p and τ be the 2-cocycle representing the class of this central extension in $\mathrm{H}^2(G, A)$ with respect to the section s. If $f : G \to G$ is an automorphism (anti-automorphism) of G, then a lift of f is an automorphism (anti-automorphism) $\tilde{f} : \widetilde{G} \to \widetilde{G}$ such that

$$p(\tilde{f}(g)) = f(p(g)), \forall g \in \widetilde{G}.$$

Let $\mathcal{L}(f)$ denote the set of all lifts of f. The group $\mathrm{Aut}(G)$ acts on $\mathrm{H}^2(G, A)$ by $f[\sigma] = [\sigma \circ (f^{-1} \times f^{-1})]$ for any 2-cocycle σ.

Proposition 2.2. *The set $\mathcal{L}(f)$ is precisely described in terms of this action by the following:*

1) *The set $\mathcal{L}(f)$ is non-empty if and only of $f[\tau] = [\tau]$.*
2) *If $\mathcal{L}(f)$ is non-empty, then $\mathcal{L}(f)$ is a principal homogeneous space for the group $\mathrm{Hom}(G, A)$ under the action*

$$(\phi.\tilde{f})(g) = \phi(p(g))\tilde{f}(g).$$

Remark 2.3. Let $G = \mathrm{GL}(2, F)$ and $\widetilde{G} = \widetilde{\mathrm{GL}}(2, F)$ be the metaplectic group with respect to $[c] \in \mathrm{H}^2(G, \mu_2)$. Let $f \in \mathrm{Aut}(G)$ be the automorphism $f(g) = w_0(g^\top)^{-1}w_o$. Since f is an involution, we have $f^{-1} = f$ and it is easy to see that $f[c]$=$[c]$. Hence there is a lift $\tilde{f}$ of f to $\widetilde{G}$.

We also need the following result (see Corollary 1 in [4] for a proof) which discusses the continuity properties of the lift in the case when $G = \mathrm{GL}(n, F)$. We state it below for clarity.

Proposition 2.4. *Let F be a non-Archimedean local field and suppose that the group of n^{th} roots of unity in F has order n. Let $\langle\, , \rangle$ be the n^{th} order Hilbert symbol on F and $\widetilde{\mathrm{GL}}(n)$ the corresponding metaplectic group. Then the lift of any topological automorphism of $\mathrm{GL}(n)$ to $\widetilde{\mathrm{GL}}(n)$ is also a topological automorphism.*

Remark 2.5. In the case when $n = 2$, clearly we have $|\mu_2| = 2$. Let $G = \mathrm{GL}(2, F)$ and $f \in \mathrm{Aut}(G)$ be the continuous automorphism of G described above. Then for the metaplectic group $\widetilde{G} = \widetilde{GL}(2, F)$, it is clear that any lift $\tilde{f}$ of f should also be a topological automorphism.

3. A property of the lifts

In this section, we explicitly describe a lift of the standard involution and discuss an important property of the lift which is crucial in adapting Tupan's topological argument to the metaplectic group. As in our earlier work, we use this lift to describe all the other lifts of the standard involution and show that they also satisfy a similar property.

Let $G = \mathrm{GL}(2, F)$ and $\widetilde{G} = \widetilde{\mathrm{GL}}(2, F)$ be the metaplectic double cover of G. Let τ be the standard involution on G.

For $\lambda \in F^\times$ and $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$, we let $u(\lambda) = \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}$ and $\Delta(g) = \det(g)$. It is easy to see that

$$\tau(g) = w_0 g^\top w_0 = u(\Delta(g)) g^{-1} u(1).$$

Let $\tilde{u}(\lambda) = (u(\lambda), 1)$. We extend Δ to $\widetilde{G}$ by $\Delta((g, \xi)) = \Delta(g)$. For $h \in \widetilde{G}$, define

$$\sigma(h) = \tilde{u}(\Delta(h)) h^{-1} \tilde{u}(1).$$

In our earlier work, we showed that σ is a lift of τ and is also an involution (see Section 4, Lemmas 4.2, 4.5 and 4.8 in [1]).

The main idea in [10] is to explicitly construct $z \in K$ satisfying $\tau(g) = zgz^{-1}$ for $g \in G$ and using it to show that for each $\lambda \geq 1$, the neighborhood gK_λ is self-conjugate under τ. We make some slight modifications and extend this idea (conjugation property) to the lift σ of τ. Further, we show that this conjugation property holds true for any lift σ_α (defined later) of τ to $\widetilde{G}$.

Theorem 3.1. *Let $\widetilde{Z} = \{z = (uI_2, \epsilon) \in \widetilde{G} \mid u \in F^\times\}$ (I_2 is the 2×2 identity matrix) and $h \in \widetilde{G}$. There exists $z \in \widetilde{Z}K^*$ such that*

$$\sigma(h) = zhz^{-1}.$$

Proof. For $1 \neq \epsilon \in \mu_2$, we know from earlier work that $\sigma(\epsilon) = \epsilon$ (see Lemma 4.3 in [1]). Hence, to compute $\sigma((g, \epsilon))$ for arbitrary $(g, \epsilon) \in \widetilde{G}$, it suffices to determine $\sigma((g, 1))$. Let $g \in G$ and $h = (g, 1) \in \widetilde{G}$. In this case, σ can be described in a more explicit way as

$$\sigma(h) = \begin{cases} (\tau(g), 1), \text{if } c = 0 \\ (\tau(g), \langle c, \Delta(g) \rangle), \text{if } c \neq 0. \end{cases}$$

We consider the following three cases

a) g is diagonal
b) g is upper triangular
c) g is neither diagonal nor upper triangular

and in each case we show that $\sigma(h) = zhz^{-1}$ for some $z \in \widetilde{Z}K^*$. We give the details of the calculations below.

Case a). Let $g = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, a, d \in F^\times$ and $k = \begin{bmatrix} 0 & \alpha \\ \beta & 0 \end{bmatrix}, \alpha, \beta \in \mathfrak{o}^\times$. Let $h = (g, 1)$ and $z = (k, s(k))$. Since $s(k) = 1$, $z \in K^* \subset \widetilde{Z}K^*$ and we have

$$zhz^{-1} = (\tau(g), c(k, g)c(kg, k^{-1})).$$

Since $\sigma(h) = (\tau(g), 1)$, it is enough to show that we can choose $\alpha, \beta \in \mathfrak{o}^\times$ such that

$$c(k, g)c(kg, k^{-1}) = 1.$$

It is easy to see that

$$c(k, g) = \langle a, \alpha \rangle \langle a, d \rangle$$

and

$$c(kg, k^{-1}) = \langle \beta, d \rangle.$$

Writing $a = \varpi^l a_1, d = \varpi^m d_1$, where $a_1, d_1 \in \mathfrak{o}^\times$ and using properties of the Hilbert symbol, we can simplify $c(k, g)c(kg, k^{-1})$ as summarized in the table below.

$val(a) = l$	$val(d) = m$	$c(k, g)c(kg, k^{-1})$
even	even	1
even	odd	$\langle a_1, \varpi \rangle \langle \beta, \varpi \rangle$
odd	even	$\langle \varpi, \alpha \rangle \langle \varpi, d_1 \rangle$
odd	odd	$\langle \varpi, \alpha \rangle \langle \varpi, \varpi \rangle \langle \varpi, d_1 \rangle \langle a_1, \varpi \rangle \langle \beta, \varpi \rangle$

If l and m are both odd and $\langle \varpi, \varpi \rangle = -1$, choose $\alpha = d_1$ and $\beta \in \mathfrak{o}^\times$ such that $\langle \beta, \varpi \rangle = -\langle a_1, \varpi \rangle$. The other cases are trivially true. Therefore, it follows that we can always choose $\alpha, \beta \in \mathfrak{o}^\times$ such that $c(k, g)c(kg, k^{-1}) = 1$ and hence the result.

Case b). Let $g = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}, a, d \in F^\times, b \in F$. Let $h = (g, 1) \in \widetilde{G}$. If $d = a$, we have $\sigma(h) = h$ and hence the result follows. Suppose that $d \neq a$. As in [10], we consider the cases

1. $val(d - a) \geq val(b)$
2. $val(a - d) < val(b)$

and in both the cases we show that there exists $z \in \widetilde{Z}K^*$ satisfying $\sigma(h) = zhz^{-1}$.

For case 1, let $k = \begin{bmatrix} 1 & 0 \\ A & 1 \end{bmatrix}$, where $A = b^{-1}(a - d)$ and $u = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \alpha \in F^\times$. Let $y = (k, s(k)) \in K^*$, $x = (u, 1) \in \widetilde{Z}$ and $z = xy \in \widetilde{Z}K^*$. We have

$$zhz^{-1} = (\tau(g), c(uk, g)c(ukg, k^{-1}u^{-1})).$$

Since $\sigma(h) = (\tau(g), 1)$, it is enough to show that we can choose $\alpha \in F^\times$ such that

$$c(uk, g)c(ukg, k^{-1}u^{-1}) = 1.$$

A simple computation shows that

$$c(uk, g) = \langle a, \alpha \rangle \langle a, A \rangle \langle a, \Delta(g) \rangle$$

and

$$c(ukg, k^{-1}u^{-1}) = \langle a, \alpha \rangle \langle \alpha, \Delta(g) \rangle \langle a, A \rangle \langle A, \Delta(g) \rangle.$$

Writing $a = \varpi^l a_1, d = \varpi^m d_1, A = \varpi^n A_1$, where $a_1, d_1, A_1 \in \mathfrak{o}^\times$ and using properties of the Hilbert symbol, we can simplify $c(uk, g)c(ukg, k^{-1}u^{-1})$ as summarized in the table below.

$val(a) = l$	$val(d) = m$	$val(A) = n$	$c(uk, g)c(ukg, k^{-1}u^{-1})$
even	even	even	$\langle \alpha, a_1 d_1 \rangle$
even	odd	even	$\langle \varpi, a_1 A_1 \rangle \langle \varpi, \alpha \rangle \langle \alpha, a_1 d_1 \rangle$
odd	even	even	$\langle \varpi, \varpi \rangle \langle \varpi, d_1 A_1 \rangle \langle \varpi, \alpha \rangle \langle \alpha, a_1 d_1 \rangle$
odd	odd	even	$\langle \varpi, a_1 d_1 \rangle \langle \alpha, a_1 d_1 \rangle$
even	even	odd	$\langle \varpi, a_1 d_1 \rangle \langle \alpha, a_1 d_1 \rangle$
even	odd	odd	$\langle \varpi, \varpi \rangle \langle \varpi, d_1 A_1 \rangle \langle \varpi, \alpha \rangle \langle \alpha, a_1 d_1 \rangle$
odd	even	odd	$\langle \varpi, a_1 A_1 \rangle \langle \varpi, \alpha \rangle \langle \alpha, a_1 d_1 \rangle$
odd	odd	odd	$\langle \alpha, a_1 d_1 \rangle$

From the above table, it is clear that we can always choose $\alpha \in F^\times$ such that $c(uk, g)c(ukg, k^{-1}u^{-1}) = 1$ in each case. Hence the result.

For case 2, let $k = \begin{bmatrix} -(\alpha\delta T) & \alpha\delta \\ \alpha\delta & -(\alpha\delta T) \end{bmatrix}$, where $\alpha \in \mathfrak{o}^\times$, $T = b(d-a)^{-1}$ and $\delta = (1-T^2)^{-1}$. Let $z = (k, 1)$. Since $val(T) = val(b) - val(d-a) > 0$, it follows that $1 - T^2 \in 1 + \mathfrak{p}$ and hence a square in $F^\times$. It follows that $s(k) = 1$ and hence $z = (k, 1) \in K^* \subset \widetilde{Z}K^*$. Also

$$zhz^{-1} = (\tau(g), c(k, g)c(kg, k^{-1})) = \langle a, d \rangle \langle \alpha, a \rangle \langle \alpha, d \rangle.$$

Since $\sigma(h) = (\tau(g), 1)$, it is enough to show that we can choose $\alpha \in \mathfrak{o}^\times$ such that

$$c(k, g)c(kg, k^{-1}) = 1.$$

Writing $a = \varpi^l a_1, d = \varpi^m d_1$, where $a_1, d_1 \in \mathfrak{o}^\times$ and using properties of the Hilbert symbol, we can simplify $c(k, g)c(kg, k^{-1})$ as summarized in the table below.

$val(a) = l$	$val(d) = m$	$c(k, g)c(kg, k^{-1})$
even	even	1
even	odd	$\langle a_1, \varpi \rangle \langle \alpha, \varpi \rangle$
odd	even	$\langle \varpi, \alpha \rangle \langle \varpi, d_1 \rangle$

It is clear that in all these cases we can always choose $\alpha \in \mathfrak{o}^\times$ such that $c(k, g)c(kg, k^{-1}) = 1$.

If $val(a) = l$ and $val(d) = m$ are both odd, then we take

$$k = \begin{bmatrix} b(a-d)^{-1} & -\alpha \\ 1 & b(a-d)^{-1} \end{bmatrix}, \quad \alpha \in \mathfrak{o}^\times.$$

Clearly $z = (k, 1) \in K^*$ and computing as before, we see that

$$c(k, g)c(kg, k^{-1}) = \langle \varpi, a_1 d_1 \rangle \langle \alpha, \varpi \rangle.$$

It follows that in all the cases, we can always choose $\alpha \in \mathfrak{o}^\times$ such that $c(k, g)c(kg, k^{-1}) = 1$ and the result follows.

Case c). Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G$, and $c \neq 0$ and $h = (g, 1) \in \widetilde{G}$. Throughout we suppose that $val(b) \geq val(c)$ (otherwise replace g by its transpose ${}^{\top}g$). As earlier, we consider the cases

1. $val(d - a) \geq val(b)$
2. $val(a - d) < val(b)$

and in both the cases we show that there exists $z \in \widetilde{Z}K^*$ satisfying $\sigma(h) = zhz^{-1}$.

For case 1, let $k = \begin{bmatrix} 1 & A \\ 0 & 1 \end{bmatrix}$, where $A = c^{-1}(d - a)$ and $u = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}, \alpha \in F^{\times}$. Let $y = (k, s(k)) \in K^*$, $x = (u, 1) \in \widetilde{Z}$ and $z = xy \in \widetilde{Z}K^*$. We have

$$zhz^{-1} = (\tau(g), c(uk, g)c(ukg, k^{-1}u^{-1})c(uk, (uk)^{-1})).$$

Since $\sigma(h) = (\tau(g), \langle c, \Delta(g) \rangle)$, it is enough to show that we can choose $\alpha \in F^{\times}$ such that

$$c(uk, g)c(ukg, k^{-1}u^{-1})c(uk, (uk)^{-1}) = \langle c, \Delta(g) \rangle.$$

Computing the relevant cocycles, we see that

$$c(ukg, k^{-1}u^{-1}) = \langle \alpha, \alpha \rangle \langle \alpha, c \rangle \langle \alpha, \Delta(g) \rangle,$$

$$c(uk, (uk)^{-1}) = \langle \alpha, \alpha \rangle,$$

$$c(uk, g) = \langle c, \alpha \rangle.$$

Therefore,

$$c(uk, g)c(ukg, k^{-1}u^{-1})c(uk, (uk)^{-1}) = \langle \alpha, \Delta(g) \rangle.$$

Choosing $\alpha = c$, the result follows.

For case 2, we assume further that $val(a) \leq val(d)$ and show that there exists $z \in \widetilde{Z}K^*$ such that $\sigma(h) = zhz^{-1}$. Let $k = \begin{bmatrix} A & -\alpha \\ \alpha & A \end{bmatrix}$, where $\alpha \in \mathfrak{o}^{\times}, A = (c\alpha + b\alpha)(a - d)^{-1}$ and let $u = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix}, \gamma \in F^{\times}$. Let $y = (k, s(k)) \in K^*$, $x = (u, 1) \in \widetilde{Z}$ and $z = xy \in \widetilde{Z}K^*$. We have

$$zhz^{-1} = (\tau(g), c(uk, g)c(ukg, k^{-1}u^{-1})).$$

Since $\sigma(h) = (\tau(g), \langle c, \Delta(g) \rangle)$, it is enough to show that we can choose $\gamma \in F^{\times}$ such that

$$c(uk, g)c(ukg, k^{-1}u^{-1}) = \langle c, \Delta(g) \rangle.$$

Let $B = a\alpha + Ac$. Computing the relevant cocycles, we see that

$$c(ukg, k^{-1}u^{-1}) = \langle c\gamma B, -c\gamma\alpha\Delta(g) \rangle,$$

$$c(uk, g) = \langle B\alpha, \gamma B\Delta(uk)c \rangle.$$

Therefore,

$$c(uk, g)c(ukg, k^{-1}u^{-1}) = \langle c, \Delta(g) \rangle \langle \gamma, \Delta(g) \rangle \langle B, \Delta(g) \rangle.$$

Since $val(a) \leq val(a - d) < val(c)$ and $val(A) > 0$, it follows that $val(B) = val(a)$. Writing $a = \varpi^l a_1$, $\Delta(g) = \varpi^m x_1$ and $B = \varpi^l B_1$, where $a_1, x_1, B_1 \in \mathfrak{o}^{\times}$ and using properties of the Hilbert symbol, we can simplify $c(uk, g)c(ukg, k^{-1}u^{-1})$ as summarized in the table below.

$val(B) = l$	$val(\Delta(g)) = m$	$c(uk, g)c(ukg, k^{-1}u^{-1})$
even	even	$\langle c, \Delta(g)\rangle\langle \gamma, x_1\rangle$
even	odd	$\langle c, \Delta(g)\rangle\langle \varpi, \gamma\rangle\langle \gamma, x_1\rangle\langle \varpi, B_1\rangle$
odd	even	$\langle c, \Delta(g)\rangle\langle \gamma, x_1\rangle\langle \varpi, x_1\rangle$
odd	odd	$\langle c, \Delta(g)\rangle\langle \varpi, \gamma\rangle\langle \gamma, x_1\rangle\langle \varpi, \varpi\rangle\langle \varpi, x_1\rangle\langle \varpi, B_1\rangle$

It is clear that we always choose $\gamma \in F^\times$ such that $c(uk, g)c(ukg, k^{-1}u^{-1}) = \langle c, \Delta(g)\rangle$. If $v(a) > v(d)$, the result follows by replacing g by $\tau(g)$. Indeed, for $r = (\tau(g), 1)$, we have all the conditions above are satisfied and hence there exists $z \in \widetilde{Z}K^*$ such that

$$\sigma(r) = (g, \langle c, \Delta(g)\rangle) = zrz^{-1}.$$

Since $\epsilon = (1, \langle c, \Delta(g)\rangle) \in \mu_2$ and $\sigma(\epsilon) = \epsilon$, the result follows. $\square$

For $\alpha \in F^\times$, let

$$\sigma_\alpha(h) = \langle \alpha, \Delta(h)\rangle\sigma(h). \tag{3.1}$$

In fact from Proposition 2.2 above, it follows that any lift of τ is of the form σ_α for $\alpha \in F^\times$. We show that all the lifts σ_α of τ also satisfy a similar conjugation property. Before we continue, we set up some notation and state a technical lemma which we need.

Lemma 3.2. *Let $h \in \widetilde{G}$ be such that $\Delta(h) \notin (F^\times)^2$. Then there exists $u \in \widetilde{Z}$ such that*

$$\epsilon h = uhu^{-1}.$$

where ϵ is the non-trivial element in μ_2.

Proof. Since $\Delta(h) \notin (F^\times)^2$, using non-degeneracy of the Hilbert symbol, it follows that there exists $\lambda \in F^\times$ such that $\langle \lambda, \Delta(h)\rangle = -1$. Let $u \in \widetilde{Z}$ be defined by

$$u = \Big(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, 1\Big).$$

A simple computation shows that

$$\epsilon h = uhu^{-1}.$$

$\square$

Theorem 3.3. *For $h \in \widetilde{G}$, we have $\sigma_\alpha(h) = zhz^{-1}$ for some $z \in \widetilde{Z}K^*$.*

Proof. Suppose $\Delta(h) \in (F^\times)^2$, then $\sigma_\alpha(h) = \sigma(h)$ and hence it follows that $\sigma_\alpha(h)$ is conjugate to h. It is enough to consider the case when $\Delta(h) \notin (F^\times)^2$ and $\langle \alpha, \Delta(h)\rangle = -1$. The result now follows from Theorem 3.1 and Lemma 3.2. For completeness, we give the details below.

$$\begin{aligned}
\sigma_\alpha(h) &= \langle \alpha, \Delta(h)\rangle\sigma(h) \\
&= \epsilon\sigma(h) \\
&= \sigma(\epsilon h) \\
&= x(\epsilon h)x^{-1} \\
&= (xu)h(xu)^{-1} \\
&= zhz^{-1}.
\end{aligned}$$

$\square$

4. Dualizing involutions on $\widetilde{G}$

In this section, we show that all the lifts σ_α of τ are dualizing involutions. Before we continue, we set up some notation and make a few observations which we need.

Let $\widetilde{H} = \{h \in \widetilde{G} \mid \Delta(h) \in (F^\times)^2\}$. It can be shown that the center of $\widetilde{H}$ is $\widetilde{Z} = \{z = (uI_2, \xi) \in \widetilde{G} \mid u \in F^\times\}$ where I_2 is the 2×2 identity matrix (see Corollary 2.13 in [2]).

Lemma 4.1. *For $\lambda \geq 1$, let $k \in K^*_\lambda$, and $z \in \widetilde{Z}$. Then*

$$zk = kz.$$

Proof. Since $char(k_F) \neq 2$, for $k \in K_\lambda$, we have $\Delta(k) \in 1 + \mathfrak{p}$ and hence a square in $F^\times$. It follows that K^*_λ is a subset of $\widetilde{H}$ for each $\lambda \geq 1$. The result follows. □

Lemma 4.2. *Let $\alpha \in F^\times$. For each $\lambda \geq 1$, we have*

$$\sigma_\alpha(K^*_\lambda) = K^*_\lambda.$$

Proof. For $x \in K_\lambda$, we have $s(x) = 1$ and hence $K^*_\lambda = \{k = (x, 1) \in \widetilde{G} \mid x \in K_\lambda\}$. Since $\Delta(x) \in (F^\times)^2$, it suffices to show that K^*_λ is invariant under σ. For $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K_\lambda$, we have

$$\sigma(k) = \begin{cases} (\tau(x), 1), \text{if } c = 0 \\ (\tau(x), \langle c, \Delta(x) \rangle), \text{if } c \neq 0. \end{cases}$$

Since K_λ is invariant under τ, the result follows. □

Lemma 4.3. *Let $\alpha \in F^\times$ and $g \in \widetilde{G}$. Then gK^*_λ and $\sigma_\alpha(gK^*_\lambda)$ are conjugate for all $\lambda \geq 1$.*

Proof. It is easy to see that for $\lambda \geq 1$, K^*_λ is a normal subgroup of K^*. Therefore, using Lemma 4.2 and Theorem 3.3, we have

$$\begin{aligned}
\sigma_\alpha(gK^*_\lambda) &= \sigma_\alpha(K^*_\lambda)\sigma_\alpha(g) \\
&= K^*_\lambda zgz^{-1} \\
&= K^*_\lambda ukgk^{-1}u^{-1}, \\
&= uK^*_\lambda kgk^{-1}u^{-1} \\
&= ukK^*_\lambda gk^{-1}u^{-1} \\
&= (ukg^{-1})gK^*_\lambda(ukg^{-1})^{-1}.
\end{aligned}$$

□

Let π be an irreducible admissible genuine representation of $\widetilde{G}$. For $f \in C^\infty_c(\widetilde{G})$, and ρ an anti-automorphism of $\widetilde{G}$, we define $f^\vee(g) = f(g^{-1})$, $f^\rho(g) = f(\rho(g))$ and $\pi^\rho(g) = \pi(\rho(g)^{-1})$.

Lemma 4.4. *For $f \in C^\infty_c(\widetilde{G})$, we have*

$$\Theta_{\pi^\vee}(f) = \Theta_\pi(f^\vee).$$

Proof. For $f \in C^\infty_c(\widetilde{G})$, it is easy to see that

$$\pi^\vee(f) = \pi^{tr}(f^\vee),$$

where $\pi^{tr}(f)$ is the transpose of the operator $\pi(f)$. Since the trace is invariant under taking transpose, it is clear that

$$\Theta_{\pi^\vee}(f) = \mathrm{Tr}(\pi^\vee(f)) = \mathrm{Tr}(\pi^{tr}(f^\vee)) = \mathrm{Tr}(\pi(f^\vee)) = \Theta_\pi(f^\vee).$$

The result follows. □

Lemma 4.5. *For $f \in C_c^\infty(\widetilde{G})$, we have*

$$\Theta_{\pi^\rho}(f) = \Theta_\pi((f^\vee)^\rho).$$

Proof. It is enough to show that $\pi^\rho(f) = \pi((f^\vee)^\rho)$. Indeed, for $v \in V$, we have

$$\begin{aligned}
\pi^\rho(f)v &= \int_{\widetilde{G}} f(g)\pi^\rho(g)vdg \\
&= \int_{\widetilde{G}} f(g)\pi(\rho(g)^{-1})vdg \\
&= \int_{\widetilde{G}} f(\rho(g)^{-1})\pi(g)vdg \\
&= \int_{\widetilde{G}} (f^\vee)^\rho(g)\pi(g)vdg \\
&= \pi((f^\vee)^\rho)v.
\end{aligned}$$

From this it follows that $\mathrm{Tr}(\pi^\rho(f)) = \mathrm{Tr}(\pi((f^\vee)^\rho))$ and hence the result. □

4.1. **Proof of the Main Theorem.** In this section, we prove the main result (Theorem 1.2) of this paper. Throughout we let $f \in C_c^\infty(\widetilde{G})$ and fix $\rho = \sigma_\alpha$ to be the anti-automorphism described in (3.1). From Lemma 4.4 and Lemma 4.5, it is enough to show that

$$\Theta_\pi(f) = \Theta_\pi(f^\rho) \tag{4.1}$$

for all $f \in C_c^\infty(\widetilde{G})$. Since f has compact support, it is enough to show that (4.1) holds for $f = \chi_{gK_\lambda^*}$, the characteristic function of gK_λ^*. Using Theorem 3.3, it follows that there exists $z \in \widetilde{G}$ such that $\rho(gK_\lambda^*) = z(gK_\lambda^*)z^{-1}$. We have

$$\begin{aligned}
\pi(f^\rho) &= \pi(\chi_{gK_\lambda^*}^\rho) \\
&= \pi(\chi_{\rho(gK_\lambda^*)}) \\
&= \pi(\chi_{z(gK_\lambda^*)z^{-1}}) \\
&= \pi(z)\pi(\chi_{gK_\lambda^*})\pi(z^{-1}) \\
&= \pi(z)\pi(f)\pi(z^{-1}).
\end{aligned}$$

Hence, $\Theta_\pi(f^\rho) = \Theta_\pi(f)$ and the result is established.

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Kumar Balasubramanian, Department of Mathematics, IISER Bhopal, Bhopal, Madhya Pradesh 462066, India
E-mail address: bkumar@iiserb.ac.in

Ekta Tiwari, Department of Mathematics, IISER Bhopal, Bhopal, Madhya Pradesh 462066, India
E-mail address: ekta@iiserb.ac.in